Extended formulation for LTA inclusion

Description, shadow prices and price formation

Version 1.2 February 2022

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Document version history

Version 1.2	February 2022	Update of Figures 1 & 2
Version 1.1	April 2020	Abridged version with minor improvements
Version 1	October 2019	Initial version

Contents

1 Introduction

We first consider in Section 2 a three nodes flow-based network example to cover concrete aspects of the LTA coverage problem and of the new methodology to perform it within Euphemia. The new methodology is based on a standard result to describe the "convex hull of the union of two polyhedra"[1].

The example is used to compare (a) models and market outcomes, (b) shadow prices and price formation, and (c) the congestion rent compared to LTA liabilities. It is shown that formulas relating bidding area market prices to shadow prices of PTDF or LTA constraints are identical to their classic flow-based or ATC counterparts. It is also discussed why the LTA coverage process guarantees to avoid a missing money issue to cover LTA liabilities.

Appendix A then presents the same developments with general notation to describe the approach and related results in full generality.

2 A detailed example

The base case example[2] is first presented, where a missing money problem for LTA liabilities occur. Next Sections then discuss approaches for the LTA coverage process.

2.1 The base case

We consider first a base case example with a three-node network (see figure 1 below) where:

- node A has two supply orders:
  400MWh @ 10€/MWh
  600MWh @ 20€/MWh
- node B has two demand orders:
  100MWh @ 70€/MWh
  900MWh @ 60€/MWh
- node C has one demand order:
  1000MWh @ 50€/MWh

and the following PTDF constraints apply:

$$0.75 netpos_b + 0.5 netpos_c \leq 250$$
$$netpos_a \geq -1500$$

Market outcome:

- A exports 450MWh, $price_a = 20$ €$/MWh$
- B imports 100MWh, $price_b = 65$ €$/MWh$
- C imports 350MWh, $price_c = 50$ €$/MWh$
- Welfare = 19 500 €
- Congestion rent $= -450 \times 20 + 100 \times 65 + 350 \times 50 = 15\ 000$€
- Order surpluses = 4 500 €

This market outcome can be obtained by solving the following welfare optimization problem (for each constraint, the associated shadow price variable is indicated on the right in square brackets, to ease later discussions):

[1] See for example Theorem 1 in [1], or Chapter 2 "Polyhera" in [2]. It is briefly formally described in Appendix.

[2] The base case is an example illustrating flow factor competition in flow-based models presented in *CWE Market Coupling, Flow-Based Forum, Amsterdam, 1st of June 2011, online:* `https://www.apxgroup.com/wp-content/uploads/Final_presentation_June_2011.pdf`.

$$\max \ welfare := (100)(70)x_{b1} + (900)(60)x_{b2} + (1000)(50)x_c - (400)(10)x_{a1} - (600)(20)x_{a2} \quad (1)$$

$$netpos_a = -400x_{a1} - 600x_{a1} \qquad [price_a = 20] \quad (2)$$
$$netpos_b = 100x_{b1} + 900x_{b2} \qquad [price_b = 65] \quad (3)$$
$$netpos_c = 1000x_c \qquad [price_c = 50] \quad (4)$$
$$0 \le x_i \le 1 \qquad \forall i \quad (5)$$

PTDF constraints:

$$0.75netpos_b + 0.5netpos_c \le 250 \qquad [ShadowPrice_1^{FB} = 60] \quad (6)$$
$$-netpos_a \le 1500 \qquad [ShadowPrice_2^{FB} = 0] \quad (7)$$

Net exports can be linked to (non-unique) commercial flows:

$$netpos_a = flow_{ba} + flow_{ca} - flow_{ab} - flow_{ac} \qquad [price_a^{FB} = 20] \quad (8)$$
$$netpos_b = flow_{ab} + flow_{ac} - flow_{ba} - flow_{bc} \qquad [price_b^{FB} = 20] \quad (9)$$
$$netpos_c = flow_{ac} + flow_{bc} - flow_{ca} - flow_{cb} \qquad [price_c^{FB} = 20] \quad (10)$$
$$f \ge 0 \quad (11)$$

Note that (8)-(10) imply :

$$netpos_a + netpos_b + netpos_c = 0 \quad (12)$$

- If we assume that the contracted volume of LTA rights is 400 MWh in the direction $A \to B$, as the price difference is $65 - 20 = 45$, the LTA liabilities are $45 \times 400 = 18000$€.
- However, the congestion rent is equal to: $-450 \times 20 + 350 \times 50 + 100 \times 65 = 15\,000$€. Hence, the congestion rent does not cover the LTA liabilities of 18 000€.

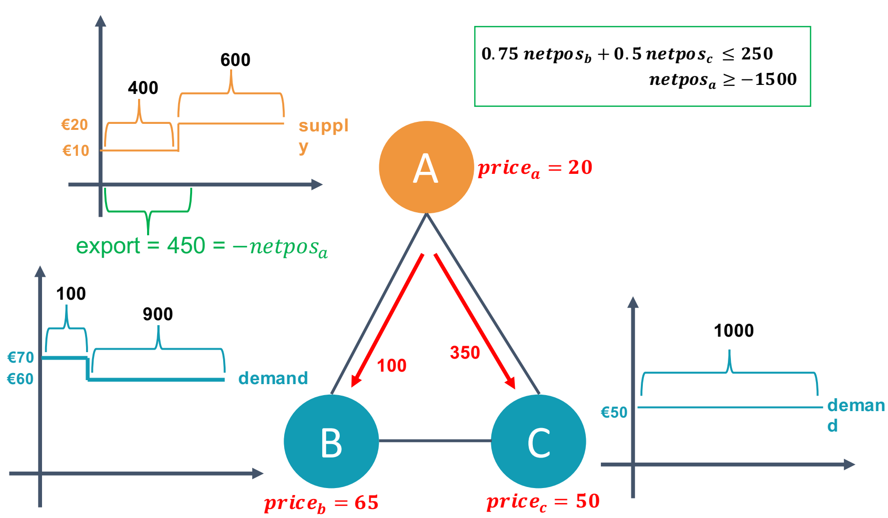

Figure 1: Base case example.

- This is related to the fact that a solution where A exports to B 400MWh and $netpos_c = 0$, i.e. where $netpos_a = -400, netpos_b = 400, netpos_c = 0$, "commercial flow" $flow_{ab} = 400$ and all other variables are null, is not feasible for the network model: constraint (6) would be violated[3].
- To solve this issue, a preferred solution is to enlarge the FB model "just enough" so as to contain that possibility $netpos_a = -400, netpos_b = 400, netpos_c = 0, flow_{ab} = 400$ (other variables null).
- This is done by considering the smallest network model that can be described by linear constraints of the form $ax \leq b$, which contains both the initial feasible points and the new possibility $netpos_a = -400, netpos_b = 400, flow_{ab} = 400$. Technically, we want the adherence of the convex hull of the union of the initial flow-based domain, and of the LTA domain(the new possibility to add), denoted $\overline{conv}(FB \cup LTA)$.[4]

2.2 The virtual branch approach for $\overline{conv}(FB \cup LTA)$

2.2.1 Model and market outcome

For the base case example above, let FB be the set of feasible net positions and flows described by conditions (6)-(11) and let LTA be the set of net positions that can be obtained by allowing a flow $f_{ab} \in [0; 400], netpos_a = -f_{ab}, netpos_b = f_{ab}, netpos_c = 0$ (and all other flows set to zero). Actually, for LTA, only the extreme case $f_{ab} = 400, netpos_a = -400, netpos_b = 400, netpos_c = 0$ (other flows null) needs to be considered for inclusion in the network model: all intermediate cases with $f_{ab} \in [0; 400]$ will automatically be included as well.

In the Virtual branch approach, one uses new PTDF constraints to describe $\overline{conv}(FB \cup LTA)$. The market outcome obtained is further discussed below and depicted on figure 2. It can be shown that for our example, the new PTDF constraints (18)-(20) together with the system condition $netpos_a + netpos_b + netpos_c = 0$ (here replaced by (21)-(23) as done in the base example above) exactly describes $\overline{conv}(FB \cup LTA)$.

The following small welfare optimization problem hence describes the market clearing problem with the network model enlarged just enough to *guarantee* that the congestion rent will cover the LTA liabilities.

$$\max\ welfare := (100)(70)x_{b1} + (900)(60)x_{b2} + (1000)(50)x_c - (400)(10)x_{a1} - (600)(20)x_{a2} \quad (13)$$

$$netpos_a = -400x_{a1} - 600x_{a1} \qquad [price_a = 20] \quad (14)$$

$$netpos_b = 100x_{b1} + 900x_{b2} \qquad [price_b = 63.75] \quad (15)$$

$$netpos_c = 1000x_c \qquad [price_c = 50] \quad (16)$$

$$0 \leq x_i \leq 1 \qquad \forall i \quad (17)$$

PTDF constraints including (VB):

[3] It is shown below that if this possibility is feasible for the network model, it is guaranteed that the congestion rent covers the LTA liabilities.

[4] Technically, one wants this convex hull "with its boundary included", hence the notation $\overline{conv}$ to distinguish from $conv$, to ensure that the enlarged set is still a polyhedron of the form $\{x|Ax \leq b\}$. This is because if P_1 and P_2 are two polyhedra, i.e. two sets of the form $\{x|Ax \leq b\}$, $conv(P_1 \cup P_2)$ might not be a polyhedron as it might "not contain its boundary", see Appendix C for a small illustrative example and discussion.

$$-2netpos_a + netpos_b \leq 1200 \qquad [ShadowPrice_1^{FB} = 0] \tag{18}$$
$$-24netpos_a + 11netpos_b \leq 14000 \qquad [ShadowPrice_2^{FB} = 1.25] \tag{19}$$
$$-netpos_a \leq 1500 \qquad [ShadowPrice_3^{FB} = 0] \tag{20}$$

Net exports can be linked to (non-unique) commercial flows,:

$$netpos_a = flow_{ba} + flow_{ca} - flow_{ab} - flow_{ac} \qquad [price_a^{FB} = 50] \tag{21}$$
$$netpos_b = flow_{ab} + flow_{ac} - flow_{ba} - flow_{bc} \qquad [price_b^{FB} = 50] \tag{22}$$
$$netpos_c = flow_{ac} + flow_{bc} - flow_{ca} - flow_{cb} \qquad [price_c^{FB} = 50] \tag{23}$$
$$f \geq 0 \tag{24}$$

Market outcome for LTA inclusion based on the virtual branch approach:

- A exports 537.5MWh, $price_a = 20$ €$/MWh$
- B imports 100MWh, $price_b = 63.75$ €$/MWh$
- C imports 437.5MWh, $price_c = 50$ €$/MWh$
- Welfare = 22 125 €
- Congestion rent $= -537.5 \times 20 + 100 \times 63.75 + 437.5 \times 50 = 17\ 500$€
- Order surpluses = 4 625 €

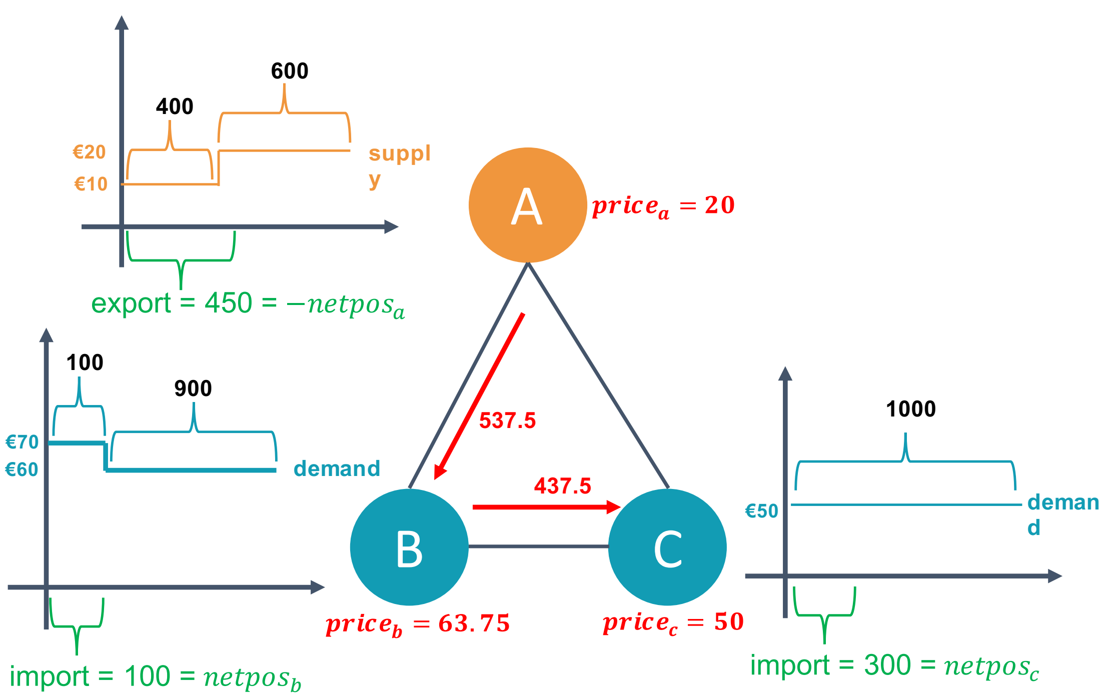

Figure 2: Market outcome with LTA coverage

2.2.2 Shadow prices and price formation

One can observe the following classical relations:

$$price_l = price^{FB} + \sum_m ptdf_{m,l} ShadowPrice_m \tag{25}$$

where $price^{FB}$ corresponds to the "system price" equal to $50 = price_a^{FB} = price_b^{FB} = price_c^{FB}$ in the example, see (13)-(24). One can derive from these relations:

$$price_k - price_l = \sum_m ShadowPrice_m (ptdf_{m,k} - ptdf_{m,l}) \tag{26}$$

For example, the price difference

$$price_b - price_a = 63.75 - 20 = 43.75 \tag{27}$$

is equal to

$$ShadowPrice_2^{FB}(ptdf_{2,b} - ptdf_{2,a}) = 1.25[11 - (-24)] = 43.75 \tag{28}$$

2.2.3 Congestion rent and LTA coverage

We observe now that the congestion rent covers the LTA liabilities:

- LTA liabilities are given by the contracted volume times the price difference between area B and area A, that is $400 \times (63.75 - 20) = 400 \times 43.75 = 17500$€
- On the other side, the congestion rent is equal also to 17500€, cf. the computation above.

The missing money problem has disappeared.

Let us see on this example the reason why the missing money disappears in general when one considers $\overline{conv}(FB \cup LTA)$ (the statement with more general notation is discussed in Section A.3 below).

This is related to the following: for the market prices $price_a^* = 20, price_b^* = 63.75, price_c^* = 50$ considered as fixed parameters, the operation of the transmission system given by the market outcome $netpos_a = -537.5, netpos_b = 100, netpos_c = 437.5$ is optimal for the following maximization problem:

$$\max congestion\ rent = 20\ netpos_a + 63.75\ netpos_b + 50\ netpos_c \tag{29}$$

subject to the network model constraints (18)-(24).

As the constraints (18)-(24) contain "by construction" the point $netpos_a = -400, netpos_b = 400, netpos_c = 0$, we know that the maximum obtained in (29) is at least equal to $20 \times (-400) + 63.75 \times 400 = 17500$€.

So whatever the solution of the market outcome is, as the net positions will be optimal for (29), the congestion rent will be at least 17 500 € (given the market prices 20, 63.75 and 50 in this example, but the same reasoning holds whatever the obtained market prices are, cf Appendix A.3 for details). In the present example, using the optimal net positions $netpos_a = -537.5, netpos_b = 100, netpos_c = 437.5$ given by the market outcome, we see that the congestion rent is actually exactly equal to 17 500 €.

2.3 The Extended formulation approach for $\overline{conv}(FB \cup LTA)$

Exactly the same market outcome as with the virtual branch approach in Section 2.2, depicted on figure 2 above, is obtained with the following model based on the new methodology proposed for LTA coverage. The only difference lies in the set of shadow prices "explaining" the (same) bidding area market prices and bidding area price differences.

2.3.1 Model and market outcome

Extended formulation approach:

$$\max\ welfare := (100)(70)x_{b1} + (900)(60)x_{b2} + (1000)(50)x_c - (400)(10)x_{a1} - (600)(20)x_{a2} \tag{30}$$

$$netpos_a = -400x_{a1} - 600x_{a1} \qquad [price_a = 20] \tag{31}$$

$$netpos_b = 100x_{b1} + 900x_{b2} \qquad [price_b = 63.75] \tag{32}$$

$$netpos_c = 1000x_c \qquad [price_c = 50] \tag{33}$$

$$0 \leq x_i \leq 1 \qquad \forall i \tag{34}$$

Virgin PTDF constraints with dedicated net export and flow variables for FB:

$$0.75netpos_b^{FB} + 0.5netpos_c^{FB} \leq \alpha_1 250 \qquad [ShadowPrice_1^{FB} = 55] \tag{35}$$

$$-netpos_a^{FB} \leq \alpha_1 1500 \qquad [ShadowPrice_2^{FB} = 2.5] \tag{36}$$

$$\tag{37}$$

$$netpos_a^{FB} = flow_{ba}^{FB} + flow_{ca}^{FB} - flow_{ab}^{FB} - flow_{ac}^{FB} \qquad [price_a^{FB} = 22.5] \tag{38}$$

$$netpos_b^{FB} = flow_{ab}^{FB} + flow_{ac}^{FB} - flow_{ba}^{FB} - flow_{bc}^{FB} \qquad [price_b^{FB} = 22.5] \tag{39}$$

$$netpos_c^{FB} = flow_{ac}^{FB} + flow_{bc}^{FB} - flow_{ca}^{FB} - flow_{cb}^{FB} \qquad [price_c^{FB} = 22.5] \tag{40}$$

$$flow^{FB} \geq 0 \tag{41}$$

LTA constraints with dedicated net export and flow variables for LTA

$$flow_{ab}^{LTA} \leq \alpha_2 400 \qquad [ShadowPrice_{ab}^{LTA} = 43.75] \quad (42)$$
$$other\ flow\ variables = 0 \quad (43)$$
$$(44)$$
$$netpos_a^{LTA} = flow_{ba}^{LTA} + flow_{ca}^{LTA} - flow_{ab}^{LTA} - flow_{ac}^{LTA} \qquad [price_a^{LTA} = 20] \quad (45)$$
$$netpos_b^{LTA} = flow_{ab}^{LTA} + flow_{ac}^{LTA} - flow_{ba}^{LTA} - flow_{bc}^{LTA} \qquad [price_b^{LTA} = 63.75] \quad (46)$$
$$netpos_c^{LTA} = flow_{ac}^{LTA} + flow_{bc}^{LTA} - flow_{ca}^{LTA} - flow_{cb}^{LTA} \qquad [price_c^{LTA} = 50] \quad (47)$$
$$flow^{LTA} \geq 0 \quad (48)$$

Constraints relating the original net export and flow variables to their duplicates used to describe respectively the virgin flow-based and LTA domains:

$$netpos_i = netpos_i^{FB} + netpos_i^{LTA} \qquad i \in \{a, b, c\} \quad (49)$$
$$flow_{ij} = flow_{ij}^{FB} + flow_{ij}^{LTA} \qquad i, j \in \{a, b, c\} \quad (50)$$
$$\alpha_1 + \alpha_2 = 1 \quad (51)$$
$$\alpha_1, \alpha_2 \geq 0 \quad (52)$$

Market outcome for LTA inclusion based on the extended formulation approach:

- A exports 537.5MWh, $price_a = 20$ €$/MWh$
- B imports 100MWh, $price_b = 63.75$ €$/MWh$
- C imports 437.5MWh, $price_c = 50$ €$/MWh$
- Welfare = 22 125 €
- Congestion rent $= -537.5 \times 20 + 100 \times 63.75 + 437.5 \times 50 = 17\ 500$€
- Order surpluses = 4 625 €

2.3.2 Shadow prices and price formation

Let us first observe that we have the same bidding area market prices but a different set of shadow prices as we now have the virgin flow-based constraints and LTA constraints (somehow "scaled" by the α) to model the network, instead of virtual branches.

However, market price differences are explained by similar relations via the shadow prices of the PTDF constraints involved, namely:

Relation identical to (25):

$$price_l = price^{FB} + \sum_m ptdf_{m,l} ShadowPrice_m \quad (53)$$

where $price^{FB}$ corresponds to the "system price" now equal to $22.5 = price_a^{FB} = price_b^{FB} = price_c^{FB}$ in the example, see (38)-(40).

For example,

$$price_c = 50 = 22.5 + 55(0.5) + 2.5(0) \quad (54)$$

We can then also derive:

Relation identical to (26):

$$price_k - price_l = \sum_m ShadowPrice_m(ptdf_{m,k} - ptdf_{m,l}) \tag{55}$$

For example, the price difference

$$price_b - price_a = 63.75 - 20 = 43.75 \tag{56}$$

is equal to

$$\begin{aligned} &ShadowPrice_1^{FB}(ptdf_{1,b} - ptdf_{1,a}) + ShadowPrice_2^{FB}(ptdf_{2,b} - ptdf_{2,a}) \\ &\qquad = 55[0.75 - 0] + 2.5[0 - (-1)] = 43.75 \end{aligned} \tag{57}$$

2.3.3 Congestion rent and LTA coverage

Discussions regarding the congestion rent would be exactly the same as the discussions in Section 2.2.3 for the virtual branch based approach. Only the constraints of the network model needs to be adapted in the optimization problem for the operation of the transmission system, which is here:

$$\max congestion\ rent = 20 netpos_a + 63.75 netpos_b + 50 netpos_c \tag{58}$$

subject to the network model constraints (35)-(52).

Conditions (35)-(52) have replaced conditions (18)-(24).

A Extended formulation for LTA inclusion in general

The extended formulation is described here with general notation for sessions with classical step bid curves. How the formulation is obtained is briefly discussed in Appendix B.

A.1 Welfare maximization model

$$\max \sum_i Q^i P^i x_i \tag{59}$$

s.t.

$$\sum_{i \in Orders(l)} Q^i x_i = netpos_l \qquad [price_l] \quad \forall l \in Locations \tag{60}$$

$$0 \leq x_i \leq 1 \qquad \forall i \tag{61}$$

$$\tag{62}$$

$$netpos_l = \widetilde{netpos}_l^{FB} + \widetilde{netpos}_l^{LTA} \qquad [\widetilde{price}_l] \tag{63}$$

$$flow_{l,k} = \widetilde{flow}_{l,k}^{FB} + \widetilde{flow}_{l,k}^{LTA} \tag{64}$$

$$\alpha_1 + \alpha_2 = 1 \qquad [\eta] \tag{65}$$

$$\alpha \geq 0 \tag{66}$$

$$\sum_l ptdf_{m,l} \widetilde{netpos}_l^{FB} \leq \alpha_1 \; RAM_m \qquad [ShadowPrice_m] \forall m \tag{67}$$

$$\widetilde{netpos}_l^{FB} = \sum_{k \neq l} \widetilde{flow}_{k,l}^{FB} - \widetilde{flow}_{l,k}^{FB} \qquad [price_l^{FB}] \; \forall l \in Locations \tag{68}$$

$$\widetilde{flow}^{FB} \geq 0 \tag{69}$$

$$\widetilde{flow}_{l,k}^{LTA} \leq \alpha_2 \; capacity_{l,k}^{LTA} \qquad [w_{l,k}^{LTA}] \qquad \forall l, k \in Locations \tag{70}$$

$$\widetilde{netpos}_l^{LTA} = \sum_{k \neq l} \widetilde{flow}_{k,l}^{LTA} - \widetilde{flow}_{l,k}^{LTA} \qquad [price_l^{LTA}] \qquad \forall l \in Locations \tag{71}$$

$$\widetilde{flow}^{LTA} \geq 0 \tag{72}$$

A.2 Shadow prices and price formation

Conditions dual to the variables $netpos_l$:

$$\widetilde{price}_l = price_l \tag{73}$$

Next conditions are written by taking (73) into account and replacing $\widetilde{price}_l$ with $price_l$.

Conditions dual to the variables $\widetilde{netpos}_l^{FB}$:

$$price_l = price_l^{FB} + \sum_m ptdf_{m,l} ShadowPrice_m \tag{74}$$

Conditions dual to the variables $\widetilde{netpos}_l^{LTA}$:

$$price_l = price_l^{LTA} \tag{75}$$

Conditions dual to the variables $\widetilde{flow^{LTA}}_{l,k} \geq 0$:

$$w_{l,k}^{LTA} \geq price_k^{LTA} - price_l^{LTA} \tag{76}$$

Using the associated complementary condition $\widetilde{flow^{LTA}}_{l,k}(w_{l,k}^{LTA} - price_k^{LTA} + price_l^{LTA}) = 0$, we have:

$$\widetilde{flow^{LTA}}_{l,k} > 0 \Rightarrow price_k^{LTA} - price_l^{LTA} = w_{l,k}^{LTA} \geq 0 \tag{77}$$

Conditions dual to the variables $\widetilde{flow}_{l,k}^{FB} \geq 0$:

$$0 \geq price_k^{FB} - price_l^{FB} \tag{78}$$

Considering (78) for all pairs l, k gives $price_k^{FB} = price_l^{FB}$ (assuming that locations form a connected component), and conditions (74) can be rewritten as:

$$price_l = price^{FB} + \sum_m ptdf_{m,l} ShadowPrice_m, \tag{79}$$

where $price^{FB}$ corresponds to the "system price".

We then have the usual relations:

$$price_k - price_l = \sum_m ShadowPrice_m (ptdf_{m,k} - ptdf_{m,l}) \tag{80}$$

A.3 Congestion rent and LTA coverage

We show here with general notation that with the LTA coverage methodology, the congestion rent is always sufficient to cover LTA liabilities.

Let us consider an optimal solution $(x^*, netpos^*, flow^*)$ to the welfare maximization problem (59)-(72) and consider the market prices $price_l^*$ obtained as optimal dual variables of (60).

We want to prove that the following inequality holds, meaning that the rent congestion is higher than the LTA liabilities:

$$\sum_l netpos_l^* \; price_l^* \geq \sum_{l,k} (price_k^* - price_l^*)^+ capacity_{l,k}^{LTA} \tag{81}$$

The congestion rent appears on the left-hand side of the inequality and is expressed as the revenue from operating the transmission network, i.e. realized from the buy/sell operations: $netpos_l^* > 0$

if a volume is sold by the operator to location l, $netpos_l^* < 0$ if a volume is bought from location l and the left-hand side represents the sum of the associated money transfers given the locational market prices $price_l^*$.

The right-hand side represents LTA liabilities. The notation $(price_k^* - price_l^*)^+$ means the price in location k, minus the price in location l, the difference being counted only of the price is higher in location k, and the difference is replaced otherwise by 0: $((price_k^* - price_l^*)^+$ is hence non-null only if the price in location k is higher than in location l, in which case $(price_k^* - price_l^*)^+$ is equal to that price difference). Multiplying $(price_k^* - price_l^*)^+$ by the volume of LTA rights in the direction $l \to k$, denoted by $capacity_{l,k}^{LTA}$, and summing up over all possibilities for l, k, provides the total LTA liabilities.

To prove (81), we will use the fact that $(netpos^*, flow^*)$ obtained from the market clearing process solves the following profit maximization problem for the transmission system, where locational market prices $price^*$ are fixed parameters and where the operator seeks to find best import/export decisions $(netpos, flow)$ given those prices and assuming "infinite market depth" (i.e. without worrying about sufficient offers or demands in the order books):

$$\max_{(netpos,flow)} \sum_l netpos_l \; price_l^* \tag{82}$$

s.t. to network constraints (63)-(72).

To prove the inequality (81), it is hence sufficient to find a solution $(netpos, flow)$ feasible for (63)-(72) such that:

$$\sum_l netpos_l \; price_l^* \geq \sum_{l,k} (price_k^* - price_l^*)^+ capacity_{l,k}^{LTA}, \tag{83}$$

as the congestion rent is at least as high as the left-hand side (since the congestion rent is an optimal value of (82)).

Such a feasible solution can be straightforwardly constructed as the network model has been enlarged for that purpose, and can be given by:

$$flow_{l,k} = flow_{l,k}^{LTA} := capacity_{l,k}^{LTA} \qquad if \;\; price_k^* > price_l^*$$

$$flow_{l,k} = flow_{l,k}^{LTA} := 0 \qquad if \;\; price_k^* \leq price_l^*$$

$$netpos_l = netpos_l^{LTA} := \sum_{k \neq l} (flow_{k,l} - flow_{l,k}) = \sum_{k \neq l} (flow_{k,l}^{LTA} - flow_{l,k}^{LTA})$$

$$\alpha_{LTA} := 1$$

$$\alpha_{FB} := 0, netpos_l^{FB} := 0, flow_{l,k}^{FB} := 0$$

B From convex combinations to the extended formulation for $\overline{conv}(FB \cup LTA)$

The set of convex combinations of two points x, y is the set of points that can be written as $\alpha_1 x + \alpha_2 y$ with $\alpha_1 + \alpha_2 = 1, \alpha_1 \geq 0, \alpha_2 \geq 0$.

It is the goal of the constraints (87)-(96) where one makes a convex combination of a point $(netpos^{FB}, flow^{FB})$ in the flow-based domain described by conditions (91)-(93) and a point in the LTA domain described by conditions (94)-(96). These constraints (87)-(96) are then transformed via a substitution of variables into the formulation (63)-(72). Details follow in next paragraphs, where cases with $\alpha_1 = 0$ or $\alpha_2 = 0$ are also discussed.

For $\alpha_1 > 0, \alpha_2 > 0, \alpha_1 + \alpha_2 = 1$, constraints (91)-(93) are fully equivalent to the original flow based constraints (just multiplied by a strictly positive number α_1) and constraints (94)-(96) are similarly fully equivalent to the original LTA constraints.

Making the substitution $\widetilde{netpos^{FB}} := \alpha_1 netpos^{FB}$, $\widetilde{flow^{FB}} := \alpha_1 flow^{FB}$, $\widetilde{netpos^{LTA}} := \alpha_2 netpos^{LTA}$, $\widetilde{flow^{LTA}} := \alpha_2 flow^{LTA}$ then exactly provides the formulation (63)-(72) in Appendix A.

Let us check that this formulation (63)-(72) also works when $\alpha_1 = 0$ or $\alpha_2 = 0$.

If $\alpha_1 = 0, \alpha_2 = 1$, assuming the PTDF polyhedron is bounded[5], only the solution $\widetilde{netpos^{FB}} = 0, \widetilde{flow^{FB}} = 0$ is feasible for (67)-(69) and we actually pick-up a point in the LTA domain, as constraints (70)-(72) with $\alpha_2 = 1$ are the original LTA constraints. Similarly, if $\alpha_1 = 1, \alpha_2 = 0$, it implies that $\widetilde{netpos^{LTA}} = 0, \widetilde{flow^{LTA}} = 0$ and we actually pick up a point in the flow-based domain, as constraints (67)-(69) with $\alpha_1 = 1$ are the original flow-based constraints.

$$\max \sum_i Q^i P^i x_i \tag{84}$$

$$\sum_{i \in Orders(l)} Q^i x_i = netpos_l \tag{85}$$

$$0 \leq x_i \leq 1 \qquad \forall i \tag{86}$$

$$netpos_l = \alpha_1 \; netpos_l^{FB} + \alpha_2 \; netpos_l^{LTA} \tag{87}$$

$$flow_{l,k} = \alpha_1 \; flow_{l,k}^{FB} + \alpha_2 \; flow_{l,k}^{LTA} \tag{88}$$

$$\alpha_1 + \alpha_2 = 1 \tag{89}$$

$$\alpha \geq 0 \tag{90}$$

$$\alpha_1 \sum_l ptdf_{m,l} \; netpos_l^{FB} \leq \alpha_1 RAM_m \tag{91}$$

$$\alpha_1 netpos_l^{FB} = \alpha_1 \sum_{k \neq l} (flow_{k,l}^{FB} - flow_{l,k}^{FB}) \tag{92}$$

$$\alpha_1 flow^{FB} \geq 0 \tag{93}$$

$$\alpha_2 flow_{l,k}^{LTA} \leq \alpha_2 capacity_{l,k}^{LTA} \tag{94}$$

$$\alpha_2 netpos_l^{LTA} = \alpha_2 \sum_{k \neq l} (flow_{k,l}^{LTA} - flow_{l,k}^{LTA}) \tag{95}$$

$$\alpha_2 flow^{LTA} \geq 0 \tag{96}$$

[5] We refer to the reference [2] for the general case where polyhedra P_1, P_2 involved in the present method to describe $\overline{conv}(P_1 \cup P_2)$ are unbounded.

C An example illustrating the difference between $conv(P_1 \cup P_2)$ and $\overline{conv}(P_1 \cup P_2)$

Consider:

- $P_1 = \{(x, y)|y = 1\}$, the blue line in the figure 3 below,
- $P_2 = \{(x, y)|x = 2, y = 2\}$, the red point in the figure 3 below.

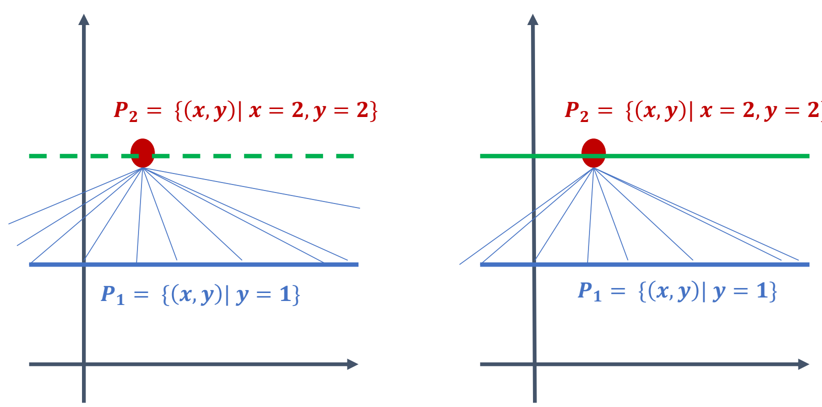

Figure 3: Difference between $conv(P_1 \cup P_2)$ and $\overline{conv}(P_1 \cup P_2)$.

One can check that all the possible convex combinations of the red point P_2 and a point in the blue line P_1 are all the points between the blue line included, and the dotted green line excluded, plus the red point: the dotted green line is a part of the boundary which is not included in "$conv(P_1 \cup P_2)$". Formally:

$$conv(P_1 \cup P_2) = \{(x, y)|(y \geq 1, y < 2)\} \cup \{(x = 2, y = 2)\} \tag{97}$$

This set cannot be described as a polyhedron, i.e. via non-strict linear inequalities.

However, if we include the boundary green line, cf. the right-hand side of figure 3, i.e. we consider "$conv(P_1 \cup P_2)$ plus its boundary included", which is written $\overline{conv}(P_1 \cup P_2)$, one has:

$$\overline{conv}(P_1 \cup P_2) = \{(x, y)|\ y \geq 1, y \leq 2\} \tag{98}$$

which is now a polyhedron.

For our optimization applications, one needs to work with the second option $\overline{conv}(P_1 \cup P_2)$.

D Bibliography

[1] Michele Conforti, Marco Di Summa, and Yuri Faenza. Balas formulation for the union of polytopes is optimal. *Mathematical Programming*, pages 1–16, 2017.

[2] Giuseppe Lancia and Paolo Serafini. *Compact extended linear programming models*. Springer, 2018.